QED at Finite Temperature in the Coulomb Gauge

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Abstract

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We argue that calculations in QED at finite temperature are more conveniently carried out in the Coulomb gauge, in which only the physical photon degrees of freedom play a rol and are thermalized. We derive the photon propagator in this gauge for real-time finite temperature calculations and show that the four-fermion static Coulomb interaction that appears in the Lagrangian can be accounted for by suitably modifying the photon propagator. The Feynman rules of the theory are written in a manifestly covariant form, although they depend on the velocity 4-vector u_μ of the background medium. As a first step in showing the consistency and usefulness of this approach, we consider the one-loop calculation of the electron self-energy Σ. It is explicitly shown that the divergences that arise from the vacuum contribution to Σ are independent of u_μ, which implies that the counter terms that must be included in the Lagrangian are the same as those in the vacuum.

1 Introduction

The real-time formulation of Finite Temperature Field Theory (FTFT) is very appealing because it can be carried out in a covariant way[1]. This, of course, does not come for free. The existence of a preferred frame when the background is a material medium, rather than the vacuum, is unavoidable and manisfests itself in the dependence of the theory on the vector u_μ representing the velocity four-vector of the background, which has components $(1, \vec{0})$ in its own rest frame. Nevertheless, the covariance of the theory thus formulated can be exploited in practical calculations, and also allows us to deduce very useful results in a general way. For example, the fact that the self-energy of a fermion propagating through a medium depends on u_μ in addition to its momentum k_μ, implies that the pole of the propagator is no longer given by the equation $k^2 = 0$, which explains why a chiral fermion will in general acquire an effective mass[2].

In a theory with fermions and scalars only, the real-time formulation of FTFT is straightforward. However, the situation is more complicated in a theory with gauge invariance like Quantum Electrodynamics (QED). At finite temperature the photon propagator in a covariant gauge (with gauge parameter λ) can be deduced in several ways[1]. A particularly instructive derivation based on a generalization of the Gupta-Bleuler quantization

method of the Stueckelberg Lagrangian[3], is summarized in Section 2. As discussed there, an essential assumption is implicitly contained in the covariant formulation; namely, that all the photon degrees of freedom are in thermal equilibrium. This is not in accord with the notion that the non-transverse photon modes do not appear in the space of states used to define the thermal averages. Motivated by a similar reasoning, Landshoff and Rebhan[4] have recently proposed a modified expression for the covariant photon propagator, in which only the transverse degrees of freedom are in thermal equilibrium.

We wish to emphasize that the origin of the difficulties lies on the insistence in formulating the theory in a covariant gauge. In the vacuum, one compelling reason for doing so is that, since the theory is Lorentz invariant, the results of any calculation should ultimately reflect that property. In particular, in non-covariant gauges an artificial dependence on some parameter analogous to the vector u_μ is introduced, which eventually disappears at the end of the calculation of a given physical quantity. Therefore, it results convenient to formulate the theory from the beginning in such a way that it is manifestly covariant at all stages of a calculation. The situation in the medium is different and more complicated. There the presence of the background sets up a preferred frame of reference. This is true even if a covariant gauge is used, and it can be appropriately accounted for by the dependence of physical observables on the vector u_μ. Therefore, there is no apparent advantage in using a covariant gauge in this case. In fact one can turn the argument around and argue that, since the theory anyway depends on u_μ, it is natural to use a gauge that also depends on this parameter. Moreover, the fact that all the unphysical photon degrees of freedom disappear in the Coulomb gauge, further suggests that this is a convenient choice for implementing QED at finite temperature.

Motivated by the above reasoning we considered in a previous work[5] the calculation and physical interpretation of the the absorptive part of the electron self-energy in the Coulomb gauge. Within the formalism of FTFT, the damping rate γ is determined from the imaginary part of the dispersion relation of the propagating mode, which is in turn obtained by looking at the pole of the corresponding propagator. In Ref. [5] a formula for γ was obtained in terms of the imaginary part of the self-energy, which is valid in the physically meaningful situation that γ and the absorptive part of the self-energy are small. It was shown there that the expression for γ coincides with the formula for the total reaction rate Γ (defined as a combination of

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probability amplitudes for various processess weighted by appropriate statistical factors)[6], provided that: (i) the amplitudes are calculated with the properly normalized spinors that satisfy the effective Dirac equation in the medium and (ii) the electron self-energy, whose imaginary part determines γ, is calculated using the Coulomb gauge for the thermal photon propagator. On the other hand, if the electron self-energy is calculated with the photon progator expressed in a general covariant gauge, then the formulas γ and Γ do not coincide. Even worse, several calculations of γ carried out during the last few years have produced contradictory results, entangled by questions of gauge invariance and other problems[7]. In that sense, the result of Ref. [5] is stimulating and suggests a way for taking a fresh look at the subject, guided by what should be a physically reasonable requirement (i.e., that $\gamma = \Gamma$) which is verified in theories where the ambiguities associated with gauge invariance are absent[5].

Eventhough we do not know whether this approach will always yield consistent results, encouraged by the above considerations we propose to adopt the Coulomb gauge as a convenient framework to do calculations in finite temperature QED. As a first consistency check, in this work we demostrate that the ultraviolet divergences of the electron self-energy do not depend on u_μ and, as a consecuence, can be substracted by means of the same counterterms as in the vacuum. Therefore, the full Lagrangian, including the counterterms, remains Lorentz invariant, despite the fact that we work in a non-covariant gauge.

In Section 2, the formula for the thermal photon propagator in a covariant gauge is deduced by adapting the Gupta-Bleuler formalism to the situation with a background. This procedure shows in a transparent form the underlying assumption that the unphysical photon degrees of freedom are in thermal equilibrium. In Section 3 we derive the Feynman rules for real-time finite temperature calculations in the Coulomb gauge, which were used in Ref. [5]. It is shown that the static Coulomb interaction, which appears at the level of the Lagrangian in this gauge, is accounted for by a suitable modification of the photon propagator. The photon propagator and the Feynman rules are written in a manifestly covariant form, although they depend on the velocity 4-vector u_μ of the background medium. In Section 4 we use those Feynman rules to calculate Σ and demonstrate that the ultraviolet divergences are independent of u_μ.

2 Photon propagator in a covariant gauge

Here we derive the finite-temperature propagator for a massive vector field by extending the Gupta-Bleuler quantization method of the Stueckelberg lagrangian[3]. As explained in the Introduction, this approach, which we have not seen in the literature, brings out some of the physical content and assumptions that lie behind the covariant formulas for the photon propagator.

Our starting point is the Stueckelberg Lagrangian,

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu - \frac{1}{2}\lambda(\partial \cdot A)^2 \,. \tag{2.1}$$

For $\lambda \neq 0$ this Lagrangian admits the correct zero-mass limit, in which we are interested. The corresponding equation of motion is

$$(\partial^2 + m^2)A^\mu - (1 - \lambda)\partial^\mu \partial \cdot A = 0 \,, \tag{2.2}$$

and taking its divergence we obtain

$$\lambda\left(\partial^2 + \frac{m^2}{\lambda}\right)\partial \cdot A = 0 \,. \tag{2.3}$$

Therefore, for non-vanishing λ, $\partial \cdot A$ is a scalar field that satisfies the Klein-Gordon Equation with square mass $M^2 = m^2/\lambda$. As a consecuence of Eq. (2.3), the field A_μ is split into two parts

$$A_\mu = A_\mu^T - \frac{\lambda}{m^2}\partial_\mu(\partial \cdot A) \,, \tag{2.4}$$

where A_μ^T is a divergenceless spin-1 vector field that satisfies the Klein-Gordon equation with mass m. Taking into account the fact that the masses of the spin-0 and spin-1 components of A_μ are different, its plane wave expansion is

$$\begin{aligned}
A_\mu(x) &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{\lambda=1}^{3} \left[a_{k\lambda}\epsilon_\mu(k,\lambda)e^{-ik\cdot x} + h.c.\right] \\
&+ \int \frac{d^3k}{(2\pi)^3 2\overline{\omega}_k} \frac{k_\mu}{m}\left[a_{k0}e^{-ik\cdot x} + h.c.\right] \,,
\end{aligned} \tag{2.5}$$

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where ω_k and $\overline{\omega}_k$ are given by

$$\begin{aligned}
\omega_k &= \sqrt{\vec{k}^2 + m^2} \\
\overline{\omega}_k &= \sqrt{\vec{k}^2 + M^2}\,,
\end{aligned} \tag{2.6}$$

and the three space-like orthonormal vectors $\epsilon_\mu(k, \lambda)$ are simultaneously orthogonal to k_μ and statisfy

$$\begin{aligned}
\epsilon(k, \lambda) \cdot \epsilon(k, \lambda') &= -\delta_{\lambda,\lambda'}\,, \\
\sum_{\lambda=1}^{3} \epsilon_\mu(k, \lambda)\epsilon_\nu(k, \lambda) &= -\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}\right).
\end{aligned} \tag{2.7}$$

Within this approach the theory is quantized by imposing the indefinite metric commutation rules

$$\begin{aligned}
[a_{k\lambda}, a^*_{k'\lambda'}] &= (2\pi)^3 2\omega_k \delta_{\lambda,\lambda'} \delta^{(3)}(\vec{k} - \vec{k}')\,, \\
[a_{k0}, a^*_{k'0}] &= -(2\pi)^3 2\omega_k \delta^{(3)}(\vec{k} - \vec{k}')\,,
\end{aligned} \tag{2.8}$$

with all the other commutators being equal to zero.

The elements of the photon propagator matrix are determined by substituting the plane wave expansion of Eq. (2.5) into the following set of relations

$$\begin{aligned}
i\Delta_{11\mu\nu}(x - y) &= \langle T\left(A_\mu(x)A_\nu(y)\right)\rangle\,, \\
i\Delta_{22\mu\nu}(x - y) &= \langle \overline{T}\left(A_\mu(x)A_\nu(y)\right)\rangle\,, \\
i\Delta_{12\mu\nu}(x - y) &= \langle A_\nu(y)A_\mu(x)\rangle\,, \\
i\Delta_{21\mu\nu}(x - y) &= \langle A_\mu(x)A_\nu(y)\rangle\,,
\end{aligned} \tag{2.9}$$

where the angle brackets denote the thermal average over the states of the system and the symbols T and $\overline{T}$ stand for the time-ordered and anti-time-ordered products. The statistical averages of the products of creation and annihilations operators are given by

$$\begin{aligned}
\langle a_{k\lambda} a^*_{k'\lambda'}\rangle &= (2\pi)^3 2\omega_k \delta_{\lambda,\lambda'} \delta^{(3)}(\vec{k} - \vec{k}')(n_k + 1)\,, \\
\langle a^*_{k\lambda} a_{k'\lambda'}\rangle &= (2\pi)^3 2\omega_k \delta_{\lambda,\lambda'} \delta^{(3)}(\vec{k} - \vec{k}')n_k\,, \\
\langle a_{k0} a^*_{k'0}\rangle &= -(2\pi)^3 2\overline{\omega}_k \delta^{(3)}(\vec{k} - \vec{k}')(\overline{n}_k + 1)\,, \\
\langle a^*_{k0} a_{k'0}\rangle &= -(2\pi)^3 2\overline{\omega}_k \delta^{(3)}(\vec{k} - \vec{k}')\overline{n}_k\,,
\end{aligned} \tag{2.10}$$

where

$$n_k = \frac{1}{e^{\beta \omega_k} - 1},$$

$$\overline{n}_k = \frac{1}{e^{\beta \overline{\omega}_k} - 1}, \tag{2.11}$$

with β denoting the inverse temperature.

Using the above the expressions, a straightforward calculation gives

$$\begin{aligned}
\Delta_{\mu\nu 11}(k) &= \frac{-g_{\mu\nu}}{k^2 - m^2 + i\epsilon} + \frac{k_\mu k_\nu}{m^2} \left[\frac{1}{k^2 - m^2 + i\epsilon} - \frac{1}{k^2 - M^2 + i\epsilon} \right] \\
&\quad - 2\pi i \eta_\gamma(k) \mathcal{O}_{\mu\nu}, \\
\Delta_{\mu\nu 21}(k) &= -2\pi i \mathcal{O}_{\mu\nu} \left[\eta_\gamma(k) + \theta(k \cdot u) \right], \\
\Delta_{\mu\nu 12}(k) &= -2\pi i \mathcal{O}_{\mu\nu} \left[\eta_\gamma(k) + \theta(-k \cdot u) \right], \\
\Delta_{\mu\nu 22}(k) &= -\Delta^*_{\mu\nu 11}(k),
\end{aligned} \tag{2.12}$$

where

$$\mathcal{O}_{\mu\nu} = -g_{\mu\nu}\delta(k^2 - m^2) + \frac{k_\mu k_\nu}{m^2} \left[\delta(k^2 - m^2) - \delta(k^2 - M^2) \right], \tag{2.13}$$

θ is the step function, and η_γ is defined by

$$\begin{aligned}
\eta_\gamma(k) &\equiv \theta(k \cdot u) n_B(x) + \theta(-k \cdot u) n_B(-x) \\
&= \frac{1}{e^{\beta|k \cdot u|} - 1}.
\end{aligned} \tag{2.14}$$

Here

$$n_B(x) = \frac{1}{e^x - 1} \tag{2.15}$$

is the boson distribution function written in terms of the variable

$$x = \beta k \cdot u. \tag{2.16}$$

The covariant expression for the photon propagator is obtained by taking the zero-mass limit in Eq. (2.12). Using the relation

$$\lim_{m^2 \to 0} \frac{1}{m^2} \left[\delta(k^2 - m^2) - \delta(k^2 - M^2) \right] = - \left(1 - \frac{1}{\lambda} \right) \frac{\partial}{\partial k^2} \delta(k^2), \tag{2.17}$$

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we get

$$
\begin{aligned}
\Delta_{\mu\nu11}(k) &= \left\{ A_{\mu\nu}(k)\frac{1}{k^2+i\epsilon} - 2\pi i\eta_\gamma(k)A_{\mu\nu}(k)\delta(k^2) \right\} , \\
\Delta_{\mu\nu21}(k) &= -2\pi i A_{\mu\nu}(k)\delta(k^2)\left[\eta_\gamma(k) + \theta(k\cdot u)\right] , \\
\Delta_{\mu\nu12}(k) &= -2\pi i A_{\mu\nu}(k)\delta(k^2)\left[\eta_\gamma(k) + \theta(-k\cdot u)\right] , \\
\Delta_{\mu\nu22}(k) &= -\Delta^*_{\mu\nu11}(k) ,
\end{aligned}
\tag{2.18}
$$

where

$$
A_{\mu\nu} = -\left[g_{\mu\nu} + \left(1 - \frac{1}{\lambda}\right) k_\mu k_\nu \frac{\partial}{\partial k^2} \right] .
\tag{2.19}
$$

The above formulas for the photon propagator coincide with the ones obtained by the time-path method[8], provided that the derivative of the delta function is interpreted according to the prescription given in Eq. (2.17). Moreover, the derivation presented here (see in particular Eq. (2.10)) exhibit in a transparent manner the fact that those formulas rest on the unjustified assumption that even the unphysical photon degrees of freedom are thermalized.

Finally, we notice that the thermal Proca propagator can be obtained directly from Eq. (2.12) in the limit $\lambda \to 0$, with $m^2 \neq 0$. This yields

$$
\begin{aligned}
\Delta_{\mu\nu11}(k) &= \Lambda_{\mu\nu}(k)\left\{ \frac{1}{k^2-m^2+i\epsilon} - 2\pi i\delta(k^2-m^2)\eta(k) \right\} , \\
\Delta_{\mu\nu21}(k) &= -2\pi i\Lambda_{\mu\nu}(k)\delta(k^2-m^2)\left[\eta(k) + \theta(k\cdot u)\right] , \\
\Delta_{\mu\nu12}(k) &= -2\pi i\Lambda_{\mu\nu}(k)\delta(k^2-m^2)\left[\eta(k) + \theta(-k\cdot u)\right] , \\
\Delta_{\mu\nu22}(k) &= -\Delta^*_{\mu\nu11}(k) ,
\end{aligned}
\tag{2.20}
$$

where

$$
\Lambda_{\mu\nu} = -g_{\mu\nu} + \frac{k_\mu k_\nu}{m^2} .
\tag{2.21}
$$

3 Photon propagator in the Coulomb gauge

The plane wave expansion of the free photon field in the Coulomb gauge is

$$
A^{tr}_\mu(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{\lambda=1,2} [a_{k\lambda}\epsilon_\mu(k,\lambda)e^{-ik\cdot x} + h.c.]
\tag{3.1}
$$

where the polarization vectors $\epsilon^\mu(k, \lambda)$ are given by

$$\epsilon^\mu(k, \lambda) = (0, \vec{e}(k, \lambda))\,,$$

with

$$\vec{e}(k, \lambda) \cdot \vec{k} = 0\,,$$

and $k^0 = \omega_k \equiv \left|\vec{k}\right|$. We have denoted the vector field by $A_\mu^{tr}(x)$ to indicate explicitly that, in this gauge, only the (physical) transverse degrees of freedom are present. The annihilation and creation operators $a_{k\lambda}$ and $a_{k\lambda}^*$ satisfy the usual commutation rules appropriate for bosons,

$$[a_{k\lambda}, a_{k'\lambda'}^*] = (2\pi)^3 2\omega_k \delta_{\lambda,\lambda'} \delta^{(3)}(\vec{k} - \vec{k}')\,. \tag{3.2}$$

The photon propagator matrix is determined from the following statistical averages

$$
\begin{aligned}
i\Delta_{11\mu\nu}^{tr}(x - y) &= \left\langle T\left(A_\mu^{tr}(x)A_\nu^{tr}(y)\right)\right\rangle, \\
i\Delta_{22\mu\nu}^{tr}(x - y) &= \left\langle \overline{T}\left(A_\mu^{tr}(x)A_\nu^{tr}(y)\right)\right\rangle, \\
i\Delta_{12\mu\nu}^{tr}(x - y) &= \left\langle A_\nu^{tr}(y)A_\mu^{tr}(x)\right\rangle, \\
i\Delta_{21\mu\nu}^{tr}(x - y) &= \left\langle A_\mu^{tr}(x)A_\nu^{tr}(y)\right\rangle,
\end{aligned}
\tag{3.3}
$$

where the symbols T and $\overline{T}$ have the same meaning as in Eq. (2.9). The calculation of the propagator is identical to the one presented in the previous section, with some obvious modifications. Thus, substituting the plane wave expansion of A_μ^{tr} in Eq. (3.3) and following steps similar to those that lead to Eq. (2.12), in the present case we obtain

$$\Delta_{ab\mu\nu}^{tr}(k) = (-R_{\mu\nu})\Delta_{ab}(k)\,, \tag{3.4}$$

where

$$
\begin{aligned}
\Delta_{11}(k) &= \frac{1}{k^2 + i\epsilon} - 2\pi i \eta_\gamma \delta(k^2)\,, \\
\Delta_{22}(k) &= \frac{-1}{k^2 - i\epsilon} - 2\pi i \eta_\gamma \delta(k^2)\,, \\
\Delta_{12}(k) &= -2\pi i \delta(k^2)[\eta_\gamma + \theta(-k \cdot u)]\,, \\
\Delta_{21}(k) &= -2\pi i \delta(k^2)[\eta_\gamma + \theta(k \cdot u)]\,,
\end{aligned}
\tag{3.5}
$$

with η_γ defined in Eq. (2.14). The tensor $R_{\mu\nu}$ is given by

$$R_{\mu\nu} \equiv -\sum_{\lambda=1,2} \epsilon_\mu(k,\lambda)\epsilon_\nu(k,\lambda)\,, \tag{3.6}$$

and its explicit expression in terms of u_μ and k_μ is

$$R_{\mu\nu} = g_{\mu\nu} + \frac{1}{\kappa^2}k_\mu k_\nu - \frac{\omega}{\kappa^2}(u_\mu k_\nu + k_\mu u_\nu) + \frac{k^2}{\kappa^2}u_\mu u_\nu\,, \tag{3.7}$$

where

$$\omega = k\cdot u\,,$$
$$\kappa = \sqrt{\omega^2 - k^2}$$

are the energy and the magnitude of the 3-momentum in the rest frame of the medium. It should be noted that the term in $R_{\mu\nu}$ depending on $u_\mu u_\nu$ disappears from the background-dependent part of the propagator because it is proportional to $k^2\delta(k^2)$. That term also disappears from the background-independent part for the reason that we explain below.

In the Coulomb gauge, the interaction Lagrangian is

$$L' = e\vec{A}^{tr}\cdot\vec{j} - e^2\int d^3x \frac{\rho(x)\rho(x')}{|\vec{x} - \vec{x}'|}\,, \tag{3.8}$$

where e is the electron charge and, as usual,

$$j_\mu = \overline{\psi}\gamma_\mu\psi\,, \tag{3.9}$$

with $\rho = j_0$. Therefore, in addition to the contribution involving the photon propagator between electron lines, there appears a static four-fermion interaction that in principle must be taken into account in any calculation. However, it is possible to absorb the effect of the four-fermion interaction into the photon propagator by the following device. The interaction in Eq. (3.8) can be thought of as being produced by the exchange of a scalar particle ϕ with the following propagator

$$i\Delta^{(\phi)}(k) = \frac{i}{\kappa^2 - i\epsilon}\,, \tag{3.10}$$

and an interaction Lagrangian with the electron of the form

$$L^{(\phi)} = -e\rho\phi \tag{3.11}$$

Therefore, instead of the interaction Lagrangian given in Eq. (3.8), for calculations we can use

$$L_{int} = -e\rho\phi + e\vec{A}^{tr} \cdot \vec{j}, \tag{3.12}$$

where, in order to reproduce the effect of the four-fermion interaction, ϕ must be assigned the following propagator

$$
\begin{aligned}
\Delta_{11}^{(\phi)}(k) &= \frac{1}{\kappa^2 - i\epsilon}, \\
\Delta_{22}^{(\phi)}(k) &= \frac{-1}{\kappa^2 + i\epsilon}, \\
\Delta_{12}^{(\phi)} = \Delta_{21}^{(\phi)} &= 0.
\end{aligned}
\tag{3.13}
$$

Eq. (3.12) can be written in a compact form by introducing the field

$$A_\mu \equiv A_\mu^{tr} + u_\mu \phi. \tag{3.14}$$

Then, instead of Eq. (3.12) and the two separate propagators $\Delta_{ab\mu\nu}^{tr}$ and $\Delta(\phi)_{ab}$, we use

$$L_{int} = -e j_\mu \cdot A^\mu, \tag{3.15}$$

together with the combined propagator

$$\Delta_{ab\mu\nu} = \Delta_{ab\mu\nu}^{tr} + u_\mu u_\nu \Delta_{ab}^{(\phi)}. \tag{3.16}$$

From the formulas in Eq. (3.4) and Eq. (3.13) we finally obtain

$$\Delta_{ab\mu\nu}(k) = (-S_{\mu\nu})\Delta_{ab}(k), \tag{3.17}$$

where $S_{\mu\nu}$ is given by

$$S_{\mu\nu} = g_{\mu\nu} + \frac{1}{\kappa^2}k_\mu k_\nu - \frac{\omega}{\kappa^2}(u_\mu k_\nu + k_\mu u_\nu) \tag{3.18}$$

It is useful to observe that, at $\omega = \kappa$, $S_{\mu\nu} = R_{\mu\nu}$ and therefore, according to Eq. (3.6), it follows that

$$S_{\mu\nu}|_{\omega=\kappa} = -\sum_{\lambda=1,2} \epsilon_\mu(k,\lambda)\epsilon_\nu(k,\lambda)\Big|_{\omega=\kappa} \tag{3.19}$$

To summarize, in the Coulomb gauge the photon propagator can be taken to be $\Delta_{ab\mu\nu}$ as given in Eq. (3.17), where the elements Δ_{ab} are specified in Eq. (3.5), and with the interaction Lagrangian of Eq. (3.15).

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4 Divergences of Σ

Now, we turn our attention to the subject of the ultraviolet divergences of the real part of the electron self-energy and the way to dispose of them in calculations done within the Coulomb gauge. Since the background dependent parts are finite, we only have to worry about the vacuum term, which is given by

$$- i\Sigma^{(0)} = (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu i S_F^{(0)}(p+k) \gamma^\nu i \Delta_{\mu\nu}^{(0)}(k) \qquad (4.1)$$

where

$$S_F^{(0)} = \frac{\not{p} + \not{k}}{(p+k)^2} \qquad (4.2)$$

and

$$\Delta_{\mu\nu}^{(0)}(k) = \left(\frac{-1}{k^2 + i\epsilon} \right) \left[g_{\mu\nu} + \frac{k_\mu k_\nu}{\kappa^2} - \frac{k \cdot u}{\kappa^2}(u_\mu k_\nu + k_\mu u_\nu) \right], \qquad (4.3)$$

with $\kappa^2 = (k.u)^2 - k^2$. The contribution produced by the term proportional to $g_{\mu\nu}$ is the usual one calculated in the Feyman gauge, and it does not depend on u_μ. Our aim is to show that the contributions from the other two terms of the photon propagator can be split into a finite part that depends on u_μ, and a divergent part that is independent of it. To do that we consider the quantity

$$I = \int \frac{d^4k}{D} \gamma^\mu \gamma^\alpha \gamma^\nu (p+k)_\alpha [k_\mu k_\nu - k \cdot u(u_\mu k_\nu + k_\mu u_\nu)], \qquad (4.4)$$

where

$$D = k^2 (p+k)^2 \kappa^2 . \qquad (4.5)$$

Using the identity

$$\gamma^\mu \gamma^\alpha \gamma^\nu = \frac{1}{4}(g^{\mu\alpha}g^{\nu\lambda} - g^{\mu\nu}g^{\alpha\lambda} + g^{\mu\lambda}g^{\nu\alpha} + i\epsilon^{\mu\alpha\nu\lambda}\gamma^5)\gamma_\lambda , \qquad (4.6)$$

the integral in Eq. (4.4) can be expressed as

$$
\begin{aligned}
I = {} & \frac{1}{4} \int \frac{d^4k}{D} \gamma^\lambda \Big\{ k_\lambda [(p+k)^2 - p^2 - 2(k \cdot u)(p \cdot u)] + p_\lambda[2(k \cdot u)^2 - k^2] \\
& - 2u_\lambda(k \cdot u)[(p+k)^2 - p^2 - p \cdot k] \Big\}
\end{aligned}
\qquad (4.7)
$$

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The terms with the factor $(p+k)^2$ produce integrals of the form

$$\int d^4 k \frac{k^\alpha}{k^2 \kappa^2} \, ,$$

which vanish upon symmetric integration, while those proportional to p^2 are finite as is easily seen by the ordinary power counting argument. The rest can be rewritten in the following way

$$I = \frac{1}{4} I^{\mu\nu}(p, u) \left\{ \not{p} [2u_\mu u_\nu - g_{\mu\nu}] + 2u_\mu [p_\nu \not{u} - \gamma_\nu (p \cdot u)] \right\} \tag{4.8}$$

with

$$I_{\mu\nu}(p, u) = \int d^4 k \frac{k_\mu k_\nu}{D} \, . \tag{4.9}$$

By considering the Taylor expansion of the integrand with respect to p, we see that the difference $I_{\mu\nu}(p, u) - I_{\mu\nu}(0, u)$ is a finite quantity. Thus, in order to examine the divergent contribution to I, in Eq. (4.8) $I^{\mu\nu}(p, u)$ can be replaced by $I^{\mu\nu}(0, u)$, which has the structure

$$I_{\mu\nu}(0, u) = a g_{\mu\nu} + b u_\mu u_\nu \, . \tag{4.10}$$

The coefficients a and b are scalar quantities and can only depend on u^2, which is equal to one. Therefore, although they are infinite, a and b do not depend on u_μ. From Eq. (4.10), we have

$$I^{\mu\nu}(0, u) u_\mu \propto u^\nu \, , \tag{4.11}$$

which implies that, when contracted with $I^{\mu\nu}(0, u) u_\mu$ the term inside the second square bracket in Eq. (4.8) vanishes identically. Then, we are left only with the divergences arising from the terms in the first square bracket of this equation. They do not vanish in general, but are proportional to

$$u^\mu u^\nu I_{\mu\nu}(0, u) \, ,$$
$$g^{\mu\nu} I_{\mu\nu}(0, u) \, ,$$

both of which are independent of u_μ as a consequence of Eq. (4.10). This complete the proof of the statement we made in the Introduction that, al least at the one-loop level, the divergences of the electron-self energy are independet of the velocity four-vector of the medium.

5   Conclussions

The use of a non-covariant gauge, such as the Coulomb gauge, is inconvenient in the vacuum because it requires that the quantization be carried out in a particular frame. In contrast, the presence of the medium defines a preferred frame which can be exploited to use a gauge in which the unphysical degrees of freedom disappear alongwith the associated question of whether they have a thermal distribution or not.

In this article we examined in detail the possible dependence on u_μ of the divergent contributions to the electron self-energy calculated within the Coulomb gauge formulation of QED at finite temperature. We explicitly showed that no such dependence ocurrs at the one-loop level and, therefore, the renormalized Lagrangian of the theory preserves its Lorentz invariant structure, despite the fact that we are using a non-covariant gauge. Taking into account the difficulties encountered in the calculations of the fermion damping in the covariant gauges, the technical result of this paper reinforces the conclusions of Ref. [5] emphasizing the use of the Coulomb gauge as an appropriate choice to carry out calculations in finite temperature QED.

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[6] This definition of Γ is taken from the work of Weldon who showed, using several one-loop examples, that it could be obtained by taking the

matrix element of the imaginary part of the self-energy between a particular set of free particle spinors. Notice, however, that the relation $\gamma = \Gamma$ obtained in Refs.[5] is valid provided that Γ is calculated with the correct spinors that satisfy the effective Dirac equation in the medium and not the the ones used by Weldon. See, H. A. Weldon, Phys. Rev. D**28**, 2007 (1983). A useful reference that explains the physical interpretation of Γ is the book by L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics*, Frontiers in Physics, Lecture Note and Reprint Series (Benjamin Cummings, Reading, 1962), p. 36.

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